Homework 10
Geometry

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Proposition 0.1 (Exercise 15-1). *Let M be a smooth manifold that is the union of two orientable open submanifolds with connected intersection. Then M is orientable.*

Proof. Let $M = A \cup B$ where A, B are orientable open submanifolds with $A \cap B$ connected. Let $\mathcal{O}^A$ be an orientation for A. If $A \cap B \neq \emptyset$, choose $p \in A \cap B$ and let $\mathcal{O}^B$ be an orientation for B such that $\mathcal{O}^B_p = \mathcal{O}^A_p$; if $A \cap B = \emptyset$, then let $\mathcal{O}^B$ be any orientation for B. Then $\mathcal{O}^A$ and $\mathcal{O}^B$ induce orientations on $A \cap B$, which is connected, so by Proposition 15.9, $\mathcal{O}^A_x = \mathcal{O}^B_x$ for $x \in A \cap B$. Then

$$\mathcal{O}^M_x = \begin{cases} \mathcal{O}^A_x & x \in A \\ \mathcal{O}^B_x & x \in B \end{cases}$$

is well defined since they agree on the overlap. It is continuous since $\mathcal{O}^A$ and $\mathcal{O}^B$ are continuous, using the Gluing Lemma. Thus $\mathcal{O}^M$ is an orientation for M. □

Corollary 0.2 (Exercise 15-1). *S^n is orientable.*

Proof. We can cover S^n with two open charts, the stereographic projection omitting the north pole and its counterpart omitting the south pole. These charts are diffeomorphisms, so $S^n \setminus \{N\}$ is orientable by pulling back the standard orientation from $\mathbb{R}^n$. Likewise, $S^n \setminus \{S\}$ is orientable. The charts have connected intersection, so by the above proposition S^n is orientable. □

Proposition 0.3 (Exercise 15-2). *Let M be a smooth n-manifold. Then TM is orientable.*

Proof. Let (U_α, ϕ_α) be a countable collection of smooth charts covering M. Let $\pi : TM \to M$ be the canonical projection, and let $(\pi^{-1}(U_\alpha), \widetilde{\phi}_\alpha)$ be the usual charts for TM (defined on page 66 of Lee). We claim that the Jacobian determinant of the transition map $\widetilde{\phi}_\beta \circ \widetilde{\phi}_\alpha^{-1}$ is positive for any α, β. Let $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ be coordinate functions for ϕ_α and ϕ_β respectively. Let p be a point in the domain of our transition map. Then we can write $v \in T_pM$ as

$$v = v^j \left. \frac{\partial}{\partial x^j} \right|_p$$

Then

$$\widetilde{\phi}_\beta \circ \widetilde{\phi}_\alpha^{-1}(x^1, \dots, x^n, v^1, \dots, v^n) = \left(y^1, \dots, y^n, v^j \frac{\partial y^1}{\partial x^j}, \dots, v^j \frac{\partial y^n}{\partial x^j} \right)$$

If we denote $\widetilde{\phi}_\beta \circ \widetilde{\phi}_\alpha^{-1}$ by F, then the Jacobian of F is

$$\begin{bmatrix} \frac{\partial F^1}{\partial x^1} & \dots & \frac{\partial F^1}{\partial x^n} & \frac{\partial F^1}{\partial v^1} & \dots & \frac{\partial F^1}{\partial v^n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial x^1} & \dots & \frac{\partial F^n}{\partial x^n} & \frac{\partial F^n}{\partial v^1} & \dots & \frac{\partial F^n}{\partial v^n} \\ \frac{\partial F^{n+1}}{\partial x^1} & \dots & \frac{\partial F^{n+1}}{\partial x^n} & \frac{\partial F^{n+1}}{\partial v^1} & \dots & \frac{\partial F^{n+1}}{\partial v^n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^{2n}}{\partial x^1} & \dots & \frac{\partial F^{2n}}{\partial x^n} & \frac{\partial F^{2n}}{\partial v^1} & \dots & \frac{\partial F^{2n}}{\partial v^n} \end{bmatrix} = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \dots & \frac{\partial y^1}{\partial x^n} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y^n}{\partial x^1} & \dots & \frac{\partial y^n}{\partial x^n} & 0 & \dots & 0 \\ 0 & \dots & 0 & \frac{\partial y^1}{\partial x^1} & \dots & \frac{\partial y^1}{\partial x^n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{\partial y^n}{\partial x^1} & \dots & \frac{\partial y^n}{\partial x^n} \end{bmatrix}$$

This is just a block diagonal matrix with the Jacobian of the transition function $\phi_\beta \circ \phi_\alpha^{-1}$ occuring twice, so the Jacobian determinant of $\widetilde{\phi}_\beta \circ \widetilde{\phi}_\alpha^{-1}$ is the square of the Jacobian determinant of $\phi_\beta \circ \phi\alpha^{-1}$. Since $\phi_\beta \circ \phi_\alpha^{-1}$ is a diffeomorphism, its Jacobian determinant is always nonzero, so the square is always positive. Thus TM has a consistently oriented smooth atlas, so by Proposition 15.6 in Lee, TM is orientable. □

Proposition 0.4 (Exercise 15-2). *Let M be a smooth n-manifold. Then T^*M is orientable.*

Proof. The argument is very similar to that for TM. Let (U_α, ϕ_α) be a countable covering of M, $\pi : T^*M \to M$ be the projection, and $(\pi^{-1}(U_\alpha), \widetilde{\phi}_\alpha)$ be the usual charts on T^*M:

$$\widetilde{\phi}_\alpha(\xi_i \, dx^i|_p) = (\phi_\alpha(p), (\xi_1, \dots, \xi_n))$$

As above, we will show that the Jacobian determinant of the transition map $\widetilde{\phi}_\beta \circ \widetilde{\phi}_\alpha^{-1}$ is always positive. Let $(x^1, \dots, x^n)$ and $(y^1, \dots, y^n)$ be coordinate functions for ϕ_α and ϕ_β respectively. Let p be a point in the domain of our transition map. Then

$$\widetilde{\phi}_\beta \circ \widetilde{\phi}_\alpha^{-1}(x^1, \dots, x^n, \xi^1, \dots, \xi^n) = \left(y^1, \dots, y^n, \xi^j \frac{\partial y^1}{\partial x^j}, \dots, \xi^j \frac{\partial y^n}{\partial x^j} \right)$$

Thus the Jacobian of $F = \widetilde{\phi}_\beta \circ \widetilde{\phi}_\alpha^{-1}$ is

$$\begin{bmatrix} \frac{\partial F^1}{\partial x^1} & \dots & \frac{\partial F^1}{\partial x^n} & \frac{\partial F^1}{\partial \xi^1} & \dots & \frac{\partial F^1}{\partial \xi^n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^n}{\partial x^1} & \dots & \frac{\partial F^n}{\partial x^n} & \frac{\partial F^n}{\partial \xi^1} & \dots & \frac{\partial F^n}{\partial \xi^n} \\ \frac{\partial F^{n+1}}{\partial x^1} & \dots & \frac{\partial F^{n+1}}{\partial x^n} & \frac{\partial F^{n+1}}{\partial \xi^1} & \dots & \frac{\partial F^{n+1}}{\partial \xi^n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F^{2n}}{\partial x^1} & \dots & \frac{\partial F^{2n}}{\partial x^n} & \frac{\partial F^{2n}}{\partial \xi^1} & \dots & \frac{\partial F^{2n}}{\partial \xi^n} \end{bmatrix} = \begin{bmatrix} \frac{\partial y^1}{\partial x^1} & \dots & \frac{\partial y^1}{\partial x^n} & 0 & \dots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial y^n}{\partial x^1} & \dots & \frac{\partial y^n}{\partial x^n} & 0 & \dots & 0 \\ 0 & \dots & 0 & \frac{\partial y^1}{\partial x^1} & \dots & \frac{\partial y^1}{\partial x^n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \frac{\partial y^n}{\partial x^1} & \dots & \frac{\partial y^n}{\partial x^n} \end{bmatrix}$$

This is just a block diagonal matrix with the Jacobian of the transition function $\phi_\beta \circ \phi_\alpha^{-1}$ occuring twice, so the Jacobian determinant of $\widetilde{\phi}_\beta \circ \widetilde{\phi}_\alpha^{-1}$ is the square of the Jacobian determinant of $\phi_\beta \circ \phi\alpha^{-1}$. Since $\phi_\beta \circ \phi_\alpha^{-1}$ is a diffeomorphism, its Jacobian determinant is always nonzero, so the square is always positive. Thus T^*M has a consistently oriented smooth atlas, so by Proposition 15.6 in Lee, T^*M is orientable. □

Proposition 0.5 (Exercise 16-2). *Let $T^2 = S^1 \times S^1 \subset \mathbb{R}^4$ denote the 2-torus, defined as*

$$\{(w, x, y, z) \in \mathbb{R}^4 : w^2 + x^2 = y^2 + z^2 = 1\}$$

with the product orientation determined by the standard orientation on S^1. Let

$$\omega = xyz\ dw \wedge dy$$

Then

$$\int_{T^2} \omega = 0$$

Proof. We parametrize T^2 by $F : (0, 2\pi) \times (0, 2\pi) \to T^2$ given by $F(\theta, \phi) = (\cos\theta, \sin\theta, \cos\phi, \sin\phi)$. Then F is an orientation preserving diffeomorphism, and the image of F contains all but a set of measure zero. Using Proposition 16.8,

$$\int_{T^2} \omega = \int_{(0,2\pi)\times(0,2\pi)} F^*\omega$$

We first compute $F^*\omega$.

$$\begin{aligned} F^*\omega = F^*(xyz\ dw \wedge dy) &= \sin\theta\cos\phi\sin\phi\ d(\cos\theta) \wedge d(\cos\phi) \\ &= \sin^2\theta\cos\phi\sin^2\phi\ d\theta \wedge d\phi \end{aligned}$$

And now we can compute the integral.

$$\int_{T^2} \omega = \int_0^{2\pi}\int_0^{2\pi} \sin^2\theta\cos\phi\sin^2\phi\ d\theta d\phi = 0$$

Since we can compute using the u-substition $u = \sin\phi$.

$$\int_0^{2\pi} \cos\phi\sin^2\phi\ d\phi = \int u^2 du = \left.\frac{\sin^3\phi}{3}\right|_0^{2\pi} = 0$$

□

Proposition 0.6 (Exercise 16-4). *Let M be a compact oriented smooth n-manifold with boundary. There is no continuous retract of M onto its boundary.*

Proof. Suppose there is a continuous retract $\psi : M \to \partial M$. Then by Theorem 6.26, ψ is homotopic to a smooth map ϕ, so there is a smooth retract $\phi : M \to \partial M$¿. (That is, ϕ is a smooth map with $\phi|_{\partial M} = \mathrm{id}|_{\partial M}$.) Let η be an orientation form on ∂M (so η is an $(n-1)$-form). Then $\phi^*\eta$ is a $(n-1)$-form on M. Let $\iota : \partial M \to M$ be the inclusion. Then by Stokes's Theorem

$$\int_M d(\phi^*\eta) = \int_{\partial M} \iota^*(\phi^*\eta) = \int_{\partial M} (\phi \circ \iota)^*\eta = \int_{\partial M} \eta > 0$$

using Proposition 16.6(c). On the other hand, since η is a top degree form on ∂M, $d\eta = 0$, so

$$\int_M d(\phi^*\eta) = \int_M \phi^*(d\eta) = 0$$

This is a contradiction, so no such smooth retract exists. Hence no continuous retract exists, since a continuous retract induces a smooth retract. □

Proposition 0.7 (Exercise 16-5)**.** *Let M, N be oriented, compact, connected, smooth manifolds and $F, G : M \to N$ be homotopic diffeomorphisms. Then F, G are either both orientation preserving or both orientation reversing.*

Proof. By Theorem 6.29, since F, G are homotopic and both are smooth, there is a smooth homotopy $H : M \times I \to N$ with $H(p, 0) = F(p)$ and $H(p, 1) = G(p)$. (Note: I denotes the unit interval $[0, 1]$). Let ω be an orientation form on N. Let $\iota : \partial(M \times I) \to M \times I$ be the inclusion. Then using Stokes's Theorem,

$$\int_{M\times I} d(H^*\omega) = \int_{\partial(M\times i)} \iota^*(H^*(\omega)) \tag{0.1}$$

First we simplify the LHS of (0.1). Since ω is an orientation form, $d\omega = 0$. Using the fact that pullback commutes with the exterior derivative,

$$\text{LHS} = \int_{M\times I} d(H^*\omega) = \int_{M\times I} H^*(d\omega) = 0$$

Now we simplify the RHS of (0.1).

$$\text{RHS} = \int_{\partial M\times I} \iota^*(H^*\omega)) = \int_{\partial(M\times I)} (H \circ \iota)^*\omega$$

Note that $\partial(M \times I)$ consists of $(M \times \{0\}) \cup (M \times \{1\})$, so we can split the integral into two. After splitting it, we rewrite by unwinding the definition of pullback.

$$\begin{aligned}
\text{RHS} = \int_{\partial(M\times I)} (H \circ \iota)^*\omega &= \int_{M\times\{0\}} (H \circ \iota)^*\omega + \int_{M\times\{1\}} (H \circ \iota)^*\omega \\
&= \int_{M\times\{0\}} \omega(H \circ \iota(p, t)) + \int_{M\times\{1\}} \omega(H \circ \iota(p, t)) \\
&= \int_{M\times\{0\}} \omega(H(p, 0)) + \int_{M\times\{1\}} \omega(H(p, 1)) \\
&= \int_{M\times\{0\}} \omega(F(p)) + \int_{M\times\{1\}} \omega(G(p)) \\
&= \int_{M\times\{0\}} F^*\omega(p, 0) + \int_{M\times\{1\}} G^*\omega(p, 1)
\end{aligned}$$

Now we would like to relate the quantities

$$\int_{M\times\{0\}} F^*\omega(p, 0) \qquad \text{and} \qquad \int_M F^*\omega(p)$$

and similarly for G. It is clear that $M \times \{0\}$ is diffeomorphic to M, so these two integrals are nearly identical, except that $M \times \{0\}$ may have a different orientation than M. Since the orientation for $M \times \{1\}$ comes from appending the outward pointing vector field $\frac{\partial}{\partial t}\big|_p$,

$M \times \{1\}$ has the same orientation as M. The orientation for $M \times \{0\}$ comes from appending the outward pointing vector field $-\frac{\partial}{\partial t}\big|_p$, so $M \times \{0\}$ has opposite orientation from M. Thus

$$\int_{M\times\{0\}} F^*\omega(p,0) = \int_{-M} F^*\omega(p) = -\int_M F^*\omega(p)$$
$$\int_{M\times\{1\}} G^*\omega(p,0) = \int_M G^*\omega(p)$$

Combining our computations for the RHS and LHS of (0.1), we have

$$\int_M F^*\omega = \int_M G^*\omega$$

From this we can easily show that F, G must both be orientation preserving or both orientation reversing. Suppose that F is orientation preserving and G is orientation reversing. Then by Proposition 16.6(d),

$$\int_M F^*\omega = \int_N \omega = -\int_M G^*\omega$$

which combined with our previous computation yields

$$\int_N \omega = 0$$

This contradicts Proposition 16.6(c), so F and G must have the same effect on the orientation.

□

Proposition 0.8 (Exercise 16-6). *Let n be an integer and let S^n be the n-sphere embedded in $\mathbb{R}^{n+1}$. The following are equivalent:*

1. *n is odd.*
2. *There exists a nowhere-vanishing smooth vector field on S^n.*
3. *There exists a continuous map $V : S^n \to S^n$ satisfying $V(x) \perp x$ (with respect to the Euclidean dot product on $\mathbb{R}^{n+1}$) for all $x \in S^n$.*
4. *The antipodal map $\alpha : S^n \to S^n$ is homotopic to Id_{S^n}.*
5. *The antipodal map $\alpha : S^n \to S^n$ is orientation preserving.*

Proof. Problem 9-4 says that there exists a nowhere-vanishing vector field on S^n for n odd, which gives (1) $\implies$ (2). Problem 15-3 says that the antipodal map is orientation preserving if and only if n is odd, which gives (1) $\iff$ (5). Problem 16-5 says that if α is homotopic to Id_{S^n} then α is orientation preserving, since Id_{S^n} is obviously orientation preserving, so this is (4) $\implies$ (5). So we just need to show (2) $\implies$ (3) and (3) $\implies$ (4), then we will have

$$(1) \implies (2) \implies (3) \implies (4) \implies (5) \implies (1)$$

which shows that all are equivalent. First we show (2) $\implies$ (3). Let Y be a nowhere-vanishing smooth vector field on S^n. Then define $V : S^n \to S^n$ by

$$V(x) = \frac{Y_x}{\|Y_x\|}$$

Since Y_x is never the zero vector, it always has nonzero magnitude, and $V(x)$ is always in S^n because $\frac{Y_x}{\|Y_x\|}$ is a unit vector by contruction. V is continuous since Y_x is a smooth vector field. Since Y_x is a vector field on S^n we have $Y_x \perp x$, and hence $V(x) \perp x$. Thus (2) $\implies$ (3).

Now we show (3) $\implies$ (4). Let α be the antipodal map and let $V : S^n \to S^n$ be a continuous map so that $V(x) \perp x$ for all $x \in S^n$. First, define $H : S^n \times [0,1] \to S^n$ by

$$H(x,t) = \frac{(1-t)\alpha(x) + tV(x)}{\|(1-t)\alpha(x) + tV(x)\|}$$

First, we need to check that the denominator is never zero. If it is, then

$$(1-t)(-x) + tV(x) = 0 \implies -x + xt + tV(x) = 0 \implies x = t(x + V(x))$$

which says that $x + V(x)$ is in the span of x. Since $V(x) \perp x$, this is impossible, so H is well-defined. H is clearly continuous, as a composition of continuous functions, and $H(x,0) = \alpha(x)$ and $H(x,1) = V(x)$, so V is homotopic to α. Now define $\widetilde{H} : S^n \times [0,1] \to S^n$ by

$$\widetilde{H}(x,t) = \frac{(1-t)x + tV(x)}{\|(1-t)x + tV(x)\|}$$

As before, if the denominator is zero,

$$(1-t)x + tV(x) = 0 \implies x - tx + tV(x) = 0 \implies tx = x + V(x)$$

which is impossible because $V(x) \perp x$. Thus $\widetilde{H}$ is a homotopy from V to Id_{S^n}. Since homotopy equivalence of maps is an equivalence relation, α is homotopic to Id_{S^n}. Thus (3) $\implies$ (4). This completes the chain of needed implications, so all the statements are equivalent. □

Corollary 0.9 (Hairy Ball Theorem)**.** *There exists a nowhere-vanishing smooth vector field on S^n if and only if n is odd.*

Proof. This is (1) $\iff$ (2) from the above proposition. □